FIELD THEORETICAL AND QUANTUM MECHANICAL DESCRIPTIONS OF COLLIDING AND NON-COLLIDING ANYONS

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ABSTRACT

I discuss two techniques that can be used in the investigation of the properties of "colliding" and "non-colliding" anyons.

1. Introduction

Several definitions of anyons[1] have been given in the literature (for a recent analysis of the different possible definitions of these planar particles, see Ref.[2]). In this paper I discuss anyons in the "boson gauge"[1,3], and therefore they are described as bosons interacting through the mediation of an abelian Chern-Simons gauge field, *i.e.* the "free anyon Hamiltonian" is (for anyons of unit mass)

$$H = \tfrac{1}{2} \sum_n \left(\mathbf{p}_n - \nu \mathbf{a}_n \right)^2 , \tag{1}$$

$$a_n^k \equiv -\epsilon^{kj} \sum_{m(\neq n)} \frac{r_n^j - r_m^j}{|\mathbf{r}_n - \mathbf{r}_m|^2} , \quad 0 < \nu < 1 ,$$

where $\mathbf{r_n} \equiv (r_n^1, r_n^2)$ is the position vector of the n-th anyon, and ν is the "statistical parameter", which characterizes the anyon statistics[1,3].

In the plane also fermions can be described as bosons interacting as in (1) with the particular choice of the statistical parameter $\nu = 1$. Ordinary planar bosons obviously correspond to (1) with $\nu = 0$. The fact that the (non-integer) values of the statistical parameter ν that correspond to anyons interpolate between the bosonic ($\nu = 0$) and the fermionic ($\nu = 1$) limit plays an important role in anyon physics, and I exploit it in the following.

Whereas for bosons (fermions) it is well understood that the wave functions do not have to (have to) vanish at the "points of overlap", which are the points of configuration space where some of the particle positions coincide, the situation is more complex in the case of anyons[4]. For conceptual simplicity, in a large majority of studies only "non-colliding anyons" (*i.e.* anyons whose wave function vanish at the points of overlap) have been considered; however, in this paper anyonic wave functions are only required to be consistent with the physical conditions that they be square integrable and diverge at most at a finite number of points, and the Hamiltonian be

self-adjoint.[*] In particular, as discussed in Refs.[5,6], these conditions imply that the wave functions[†]describing the relative motion of two anyons must satisfy the following boundary condition at the point of overlap ($r = 0$)

$$\left[r^{|\nu|}\psi(\mathbf{r}) - wR^{2|\nu|}\frac{d\left(r^{|\nu|}\psi(\mathbf{r})\right)}{d(r^{2|\nu|})} \right]_{r=0} = 0 \ , \tag{2}$$

which can be equivalently expressed as the following requirement on the form of ψ for $r \sim 0$

$$\psi(\mathbf{r}) \to a(r^{|\nu|} + wR^{2|\nu|}r^{-|\nu|}) \quad \text{for } r \sim 0 \ . \tag{3}$$

R, the "self-adjoint extension scale"[‡] is a reference scale with dimensions of a length, w, the "self-adjoint extension parameter", is a dimensionless real parameter which characterizes the type of boundary condition, and a is a constant. Note at the "critical points" $w=0$, which corresponds to the conventional non-colliding anyons, and $w \sim \infty$ the theory becomes scale invariant (*i.e.* it is independent of the scale R).

I am now ready to give more rigorously the definition of anyons adopted in this paper; they are particles described by wave functions satisfying boundary conditions of the type (2), whose free evolution is governed by H given in (1). (Obviously in presence of interactions described by a potential V the evolution is governed by the Hamiltonian $H+V$.) Given a self-adjoint extension scale R the anyons defined in this way are characterized by two numbers, the statistical parameter ν and the self-adjoint extension parameter w. In the following I discuss a field theoretical and a quantum mechanical method of investigation of the properties of these particles.

2. Non-Relativistic Field Theory

In this section, I discuss a field theoretical method of investigation of anyons. Let me start by considering the Lagrange density

$$\mathcal{L} = \frac{1}{4\pi\rho}\epsilon^{\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma + i\phi^\dagger D_t\phi - \frac{1}{2}(\mathbf{D}\phi)^\dagger \cdot \mathbf{D}\phi \ , \tag{4}$$

where ϕ is a complex bosonic field, $(\mathbf{A}, A_0)$ is an auxiliary[7] abelian Chern-Simons gauge field, and $D_t \equiv \partial_t + iA_0$ and $\mathbf{D} \equiv \nabla - i\mathbf{A}$ are the covariant derivatives.

In Refs.[8,9] it was shown that renormalizability requires that a contact term $-\pi g_b(\phi^\dagger\phi)^2$ be added to $\mathcal{L}$. The renormalized s-wave two-particle scattering amplitude

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[*]In the case of free bosons and free fermions these physical conditions are sufficient to determine[4] the behavior of the wave functions at the points of overlap.

[†]Note that one can easily see[5] that for non-s-wave functions square integrability is only consistent with the $\psi(0) = 0$ boundary condition; therefore, these generalized boundary conditions only affect the s-wave part of calculations.

[‡]As clarified in Refs.[5,6], this discussion of the boundary conditions for anyons is an application of the method of self-adjoint extension of the Schrödinger Hamiltonian.

was calculated to two-loop order in Ref.[6]; including the appropriate kinematic factor it can be written as

$$A_{s,2-\text{loop}} = -\sqrt{\tfrac{2\pi}{p}}\{g_r - \tfrac{i\pi}{2}\rho^2 - \tfrac{\pi^2}{6}g_r\rho^2 + (g_r^2 - \rho^2)\left(\ln\tfrac{p}{\mu} - \tfrac{i\pi}{2}\right) + g_r(g_r^2 - \rho^2)\left(\ln\tfrac{p}{\mu} - \tfrac{i\pi}{2}\right)^2\}, \quad (5)$$

where $\mathbf{p}$ is the relative momentum, ϵ is the usual cut-off used in dimensional regularization, μ is the renormalization scale, and g_r is the two-loop renormalized contact coupling which is defined in terms of the bare contact coupling g_b by the relation (γ denotes the Euler constant)

$$g_r = g_b - (g_b^2 - \rho^2)(\tfrac{1}{2\epsilon} - \tfrac{\gamma - \ln 4\pi}{2}) + g_b(g_b^2 - \rho^2)(\tfrac{1}{2\epsilon} - \tfrac{\gamma - \ln 4\pi}{2})^2 \ , \quad (6)$$

Note that only at the critical values $g_r = \pm\rho$ of the renormalized contact coupling the scale invariance of the classical theory is preserved at the quantum level.

In order to show how to use the results of this field theory in the study of anyons, let me compare Eq.(5) to the exact s-wave scattering amplitude of anyons. This scattering amplitude can be evaluated exactly by using a rather straightforward generalization of the analysis given in Ref.[10], which concerned the special case $w = 0$; one finds[6] that

$$A_s(p) = -i\sqrt{\tfrac{2}{\pi p}}(e^{i\pi|\nu|} - 1)\frac{1 - \tfrac{1}{w}\left(\tfrac{2}{pR}\right)^{2|\nu|}\frac{\Gamma(1+|\nu|)}{\Gamma(1-|\nu|)}}{1 + \tfrac{1}{w}e^{i\pi|\nu|}\left(\tfrac{2}{pR}\right)^{2|\nu|}\frac{\Gamma(1+|\nu|)}{\Gamma(1-|\nu|)}}$$

$$= -\sqrt{\tfrac{2\pi}{p}}\{|\nu|\tfrac{1-w}{1+w} - \tfrac{i\pi}{2}\nu^2 - \nu^2\tfrac{4w}{(1+w)^2}\left(\ln\tfrac{pR}{2} + \gamma - \tfrac{i\pi}{2}\right)$$

$$- \tfrac{\pi^2}{6}|\nu|^3\tfrac{1-w}{1+w} - |\nu|^3\tfrac{4(1-w)w}{(1+w)^3}\left(\ln\tfrac{pR}{2} + \gamma - \tfrac{i\pi}{2}\right)^2 + O(\nu^4)\} \ . \quad (7)$$

If one uses the relations

$$\rho = \nu \ , \qquad g_r = |\nu|\frac{1-w}{1+w} \ , \quad (8)$$

$$\mu = \frac{2}{Re^\gamma} \quad (9)$$

the two-loop field theoretical result (5) reproduces exactly the $O(\nu^3)$ approximation of the exact result (7). This indicates that the field theory discussed in this section describes anyons of statistical parameter ρ and self-adjoint extension parameter $(|\rho| - g_r)/(|\rho| + g_r)$ at a self-adjoint extension scale $2\mu^{-1}e^{-\gamma}$.

Further insight into the correspondence between the anyon quantities w,R and the field theoretical quantities g_r,μ can be gained from the following observations. First, notice that, using the renormalization-group equation which states that the scattering amplitude obtained in field theory is independent of the choice of the renormalization scale μ, one can derive the following beta function for the coupling g_r

$$\beta(g_r) \equiv \frac{dg_r}{d\ln\mu} = g_r^2 - \rho^2 \ . \quad (10)$$

Eq.(10), which indicates that g_r and μ are not physically independent, can be integrated to give the relation

$$\frac{|\rho| + g_r(\mu_1)}{|\rho| - g_r(\mu_1)} \mu_1^{2|\rho|} = \frac{|\rho| + g_r(\mu_2)}{|\rho| - g_r(\mu_2)} \mu_2^{2|\rho|} . \qquad (11)$$

Similarly in the exact result (7) R is only a reference scale, and obviously physics must be independent of the choice of R. Indeed, all physical quantities (see, for example, Eqs.(2) and (7)) depend on w and R only through the product $wR^{2|\nu|}$, and the independence of physics on the choice of R is in the fact that if R is changed from a value R_1 to a value R_2 this must be accompanied by a corresponding change of w as described by the relation

$$w(R_1) R_1^{2|\nu|} = w(R_2) R_2^{2|\nu|} . \qquad (12)$$

Clearly, the Eqs.(11) and (12) are perfectly consistent with the relations (8) and (9).

3. Quantum Mechanics

The ideas discussed in the preceding section are also useful in quantum mechanics, leading to a quantum mechanical approach to the study of anyons. For simplicity in this section I discuss this approach only in the case of the conventional non-colliding anyons ($w = 0$), which allows a scale invariant analysis. However, one can straightforwardly verify that, like in the field theory case, the approach can be generalized to the study of "colliding anyons" ($w \neq 0$).

Let me start by considering the Hamiltonian

$$H_2 = H_2^{free} + r^2 = -\frac{1}{r}\partial_r(r\partial_r) - \frac{1}{r^2}\partial_\phi^2 + r^2 - \frac{2i\rho}{r^2}\partial_\phi + \frac{\rho^2}{r^2} . \qquad (13)$$

H_2^{free} is the two-body relative motion Hamiltonian of the quantum mechanical formulation[7] of the non-relativistic field theory with Lagrangian $\mathcal{L}$ (see Eq.(4)) discussed in the preceding section. The r^2 term describes an harmonic interaction and discretizes the spectrum.

In the perturbative calculations about $\rho = 0$, one runs into an inconsistency because the matrix elements of $\frac{\rho^2}{r^2}$ between 0-th order in ρ s-wave functions (*i.e.* s-wave functions of bosons in an harmonic potential) are logarithmically divergent; for example, to second order in ρ, one of the contributions to the energy of the state with $n = 2$ and $l = 0$ is given by (L_n^x are Laguerre polynomials)

$$< \Psi_{2,0}^{(0)}|\frac{\rho^2}{r^2}|\Psi_{2,0}^{(0)} > = \int_0^\infty \int_0^{2\pi} r \, dr \, d\phi \, \frac{e^{-\frac{r^2}{2}}}{\pi^{\frac{1}{2}}} L_2^0(r^2) \, \frac{\rho^2}{r^2} \, \frac{e^{-\frac{r^2}{2}}}{\pi^{\frac{1}{2}}} L_2^0(r^2)$$

$$= 2\rho^2 \int_0^\infty \frac{\exp(-r^2)}{r} \, dr \, \sim \infty . \qquad (14)$$

One can easily see[11-15] that these divergences are closely related to the ultraviolet divergences encountered in the field theoretical calculations based on the Lagrangian of Eq.(4), and it is therefore not surprising that they also can be cured by introducing a contact interaction[11-15]. The renormalizable Hamiltonian is

$$H_2^{ren} \equiv -\frac{1}{r}\partial_r(r\partial_r) - \frac{1}{r^2}\partial_\phi^2 + r^2 - \frac{2i\rho}{r^2}\partial_\phi + 2\pi g_b\delta^{(2)}(\mathbf{r}) + \frac{\rho^2}{r^2} = H_2 + 2\pi g_b\delta^{(2)}(\mathbf{r}) \ , \quad (15)$$

where $2\pi g_b\delta^{(2)}(\mathbf{r})$ is the two-body quantum mechanical counterpart of the $-\pi g_b(\phi^\dagger\phi)^2$ contact term of the field theory discussed in the preceding section.

Unlike H_2, H_2^{ren} is suitable for perturbation theory; in fact, with an appropriate choice of the contact coupling (which corresponds to the identifications (8) and (9)), the added δ-function potential leads to divergencies which cancel those introduced by the ρ^2/r^2 term.

Let me look at some calculations for the state with $n = 2$ and $l = 0$, which will also illustrate this mechanism of cancellation of divergencies. The first order energy is given by

$$\begin{aligned}
E_{2,0}^{(1)} &= \ <\Psi_{2,0}^{(0)}| - \frac{2i\rho}{r^2}\partial_\phi + 2\pi g_b\delta^{(2)}(\mathbf{r})|\Psi_{2,0}^{(0)}> \\
&= \ <\Psi_{2,0}^{(0)}|2\pi g_b\delta^{(2)}(\mathbf{r})|\Psi_{2,0}^{(0)}> = 2g_b \ .
\end{aligned} \quad (16)$$

Concerning the first order eigenfunction one easily finds

$$\begin{aligned}
|\Psi_{2,0}^{(1)}> &= \sum_{m,l\neq 2,0} \frac{<\Psi_{m,l}^{(0)}| -\frac{2i\rho}{r^2}\partial_\phi + 2\pi g_b\delta^{(2)}(\mathbf{r})\ |\Psi_{2,0}^{(0)}>}{E_{2,0}^{(0)} - E_{m,l}^{(0)}}|\Psi_{m,l}^{(0)}> \\
&= \sum_{m\neq 2} \frac{<\Psi_{m,0}^{(0)}|\ 2\pi g_b\delta^{(2)}(\mathbf{r})\ |\Psi_{2,0}^{(0)}>}{E_{2,0}^{(0)} - E_{m,0}^{(0)}}|\Psi_{m,0}^{(0)}> \\
&= -\frac{g_b}{2\sqrt{\pi}}\sum_{m\neq 2}\frac{L_m^0(r^2)}{m-2}e^{-\frac{r^2}{2}} \\
&= \frac{g_b}{\sqrt{\pi}}e^{-\frac{r^2}{2}}\left[\frac{3}{2} - r^2 + \frac{1}{4}(2\gamma - 3 + 4ln(r))L_2^0(r^2)\right] \ .
\end{aligned} \quad (17)$$

The second order energy is given by

$$E_{2,0}^{(2)} = E_{2,0}^{(2,a)} + E_{2,0}^{(2,b)} \ , \quad (18)$$

$$\begin{aligned}
E_{2,0}^{(2,a)} &= \ <\Psi_{2,0}^{(0)}|\ \frac{\rho^2}{r^2}\ |\Psi_{2,0}^{(0)}> = \rho^2\int_0^\infty\int_0^{2\pi} dr\ d\phi\ \frac{\exp(-r^2)}{\pi r}\ [L_2^0(r^2)]^2 \\
&= \lim_{\epsilon\to 0}\int_\epsilon^\infty dr\ \rho^2\ \frac{\exp(-r^2)}{r}\ [L_2^0(r^2)]^2 = \rho^2\lim_{\epsilon\to 0}\left[-2\ln(\epsilon) - \gamma - \frac{3}{2}\right], \quad (19)
\end{aligned}$$

$$E_{2,0}^{(2,b)} = \; < \Psi_{2,0}^{(0)}| \; -\frac{2i\rho}{r^2}\partial_\phi + 2\pi g_b \delta^{(2)}(\mathbf{r}) \; |\Psi_{2,0}^{(1)}> \; = \; < \Psi_{2,0}^{(0)}| \; 2\pi g_b \delta^{(2)}(\mathbf{r}) \; |\Psi_{2,0}^{(1)}>$$

$$= 2g_b^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dr_x \, dr_y \, \delta^{(2)}(\mathbf{r}) \, e^{-r^2} \, L_2^0(r^2) \left[\frac{3}{2} - r^2 + \frac{1}{4}(2\gamma - 3 + 4 ln(r)) L_2^0(r^2)\right]$$

$$= g_b^2 \lim_{\epsilon\to0}\left[2 ln(\epsilon) + \frac{3}{2} + \gamma\right] . \tag{20}$$

Note that I introduced a cut-off ϵ (which must be ultimately removed by taking the limit $\epsilon \to 0$) in order to see the cancellation of infinities and evaluate the left-over finite result. In general a similar cut-off must be introduced in all the divergent matrix elements of r^{-2} and $\delta^{(2)}(\mathbf{r})$ by using

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dr_x \, dr_y \frac{1}{r^2} \, f(r_x, r_y) = \lim_{\epsilon\to0}\int_{\epsilon}^{\infty}\int_{0}^{2\pi} r \, dr \, d\phi \, \frac{1}{r^2} \, f(r\cos\phi, r\sin\phi) , \tag{21}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dr_x \, dr_y \delta^{(2)}(\mathbf{r}) \, f(r_x, r_y) = \lim_{\epsilon\to0} f(\frac{\epsilon}{\sqrt{2}}, \frac{\epsilon}{\sqrt{2}}) . \tag{22}$$

I am now ready to check that the quantum mechanical approach discussed in this section can describe non-colliding ($w = 0$) anyons. Indeed the problem of two non-colliding anyons in an harmonic potential has been solved[1], and in particular it has been found that

$$E_{2,0} = (10 + 2|\nu|) , \tag{23}$$

$$|\Psi_{2,0}> = N_{2,0} \, r^{|\nu|} \, e^{-\frac{r^2}{2}+il\phi} \, L_2^{|\nu|}(r^2) , \tag{24}$$

where $N_{2,0}$ is a normalization constant.

It is easy to verify that Eqs.(23) and (24) are in perfect agreement with Eqs.(16), (17), (18), (19), and (20) once the identifications (8) are taken into account.

Note that, as shown in Ref.[14], the results obtained in this section for the two-body Schrödinger problem can be easily generalized to N-body Schrödinger problems; one finds that if H_N is the original Hamiltonian, a renormalizable perturbation theory requires the use of the Hamiltonian H_N^{ren}, given by

$$H_N^\delta \equiv H_N + 2\pi g_b \sum_{m<n} \delta^{(2)}(\mathbf{r_n} - \mathbf{r_m}) . \tag{25}$$

Also note that my choice of discussing eigenenergies and eigenfunctions in this section instead of the scattering amplitudes considered in the preceding section was only an expedient to show the general validity of these techniques. Indeed, as indicated by results of Ref.[12], also the two-anyon s-wave scattering amplitude can be calculated using this renormalized perturbation theory in quantum mechanics.

4. Conclusion

I have discussed a field theoretical and a quantum mechanical technique for the investigation of colliding ($w \neq 0$) and non-colliding ($w = 0$) anyons.

The results presented here are also relevant to the issue of which boundary conditions at the points of overlap are most natural in the case of anyons[4,16]. Indeed, the results of Sec.2 and 3 establish descriptions of colliding anyons[4], *i.e.* they identify the strength of the contact coupling g_r which (at a given scale) is to be used in the calculations in order to describe anyons whose wave functions satisfy the non-conventional boundary conditions corresponding to self-adjoint extension parameter w. We have seen that in the field theoretical approach (and one can show that this holds also for the quantum mechanical approach) there appears to be no reason for restricting oneself to the case of the conventional non-colliding anyons,[§] the ones whose wave functions vanish at the points of overlap.

As indicated by results presented in Ref.[6], the techniques here discussed for the investigation of anyons can be rather straightforwardly generalized to the case of "non-abelian anyons" (also called "NACS particles"[6,17]), which are particles that can be described as bosons interacting through the mediation of a non-abelian Chern-Simons gauge field.

Acknowledgements

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[§]The only special property of the non-colliding anyons, which correspond to $w = 0$, is the preservation of the scale invariance[6]; however, this property is shared by the case of colliding anyons with $w \sim \infty$, and, anyway, there appears to be no physical motivation for excluding values of w that do not preserve scale invariance.

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